PROPER HOLOMORPHIC POLYNOMIAL MAPS BETWEEN BOUNDED SYMMETRIC DOMAINS OF CLASSICAL TYPE

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ABSTRACT. We prove that two proper holomorphic polynomial maps between bounded symmetric domains of classical type which preserve the origin are equivalent if and only if they are isotropically equivalent. Using this property we show that each member of a one-parameter family of maps from [19] is inequivalent.

1. INTRODUCTION

Let Ω_1, Ω_2 be domains in $\mathbb{C}^n$ and $\mathbb{C}^N$ and $f, g : \Omega_1 \to \Omega_2$ be holomorphic maps. We say that f is *proper* if $f^{-1}(K)$ is compact for every compact subset $K \subset \Omega_2$. We say that f and g are *equivalent* if and only if $f = A \circ g \circ B$ for some $B \in \mathrm{Aut}(\Omega_1)$ and $A \in \mathrm{Aut}(\Omega_2)$. For a domain Ω, denote the group of automorphisms fixing $p \in \Omega$ by $\mathrm{Isot}_p(\Omega)$. Suppose that for fixed $p \in \Omega_1$, $f(p) = g(p)$. Then we say that f and g are *isotropically equivalent at p* if there are $U \in \mathrm{Isot}_p(\Omega_1)$ and $V \in \mathrm{Isot}_{g(p)}(\Omega_2)$ such that $f = V \circ g \circ U$. The notion of isotropic equivalence coincides with that of unitary equivalence of [3] defined when Ω_1 and Ω_2 are balls. The following domains are called bounded symmetric domains of classical type:

(1) $\Omega_{r,s}^I = \{Z \in M_{r,s}^{\mathbb{C}} : I_r - ZZ^* > 0\}$, where $s \geq r = \mathrm{rank}(\Omega_{r,s}^I)$.

(2) $\Omega_n^{II} = \{Z \in M_{n,n}^{\mathbb{C}} : I_n - ZZ^* > 0, \ Z^t = -Z\}$, $\mathrm{rank}(\Omega_n^{II}) = \left[\frac{n}{2}\right]$.

(3) $\Omega_n^{III} = \{Z \in M_{n,n}^{\mathbb{C}} : I_n - ZZ^* > 0, \ Z^t = Z\}$, $\mathrm{rank}(\Omega_n^{III}) = n$.

(4) $\Omega_n^{IV} = \left\{Z = (z_1, \ldots, z_n) \in \mathbb{C}^n : ZZ^* < 1, 0 < 1 - 2ZZ^* + \left|ZZ^t\right|^2\right\}$, $\mathrm{rank}(\Omega^{IV}) = 2$.

Here we denote by $M > 0$ positive definiteness of square matrix M, by $M_{r,s}^{\mathbb{C}}$ the set of $r \times s$ complex matrices and by I_r the $r \times r$ identity matrix.

The aim of this paper is to prove the following theorems as a generalization of the results in [3] which are concerned with the proper holomorphic polynomial maps between balls.

Theorem 1.1. *Let Ω_1, Ω_2 be bounded symmetric domains of classical type and $f, g : \Omega_1 \to \Omega_2$ proper holomorphic polynomial maps such that $f(0) = g(0) = 0$. Then f and g are equivalent if and only if they are isotropically equivalent at 0.*

Theorem 1.2. *There are uncountably many inequivalent proper holomorphic maps from $\Omega_{r,s}^I$ to $\Omega_{2r-1,2s}^I$ for $r \geq 2$, $s \geq 2$.*

The motivation of this paper comes from generalizing the study on proper holomorphic maps between balls to bounded symmetric domains of rank greater than or equal to two. Proper holomorphic maps between balls have been studied for a long time since Alexander ([1]) proved that every proper holomorphic self-map of the n-dimensional unit ball $\mathbb{B}^n$ with $n \geq 2$ is a holomorphic automorphism. For proper holomorphic maps between balls with different dimensions, much work has been done, relating the maximum degree of proper holomorphic maps to the difference of dimensions between the domain ball and the target ball. As the first work along these lines, Webter ([22]) proved that any proper holomorphic map from $\mathbb{B}^n$ to $\mathbb{B}^{n+1}$ with $n \geq 3$, C^3-smooth up to the boundary, is equivalent to the embedding

$$f_s : (z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_n, 0).$$

Given a proper holomorphic map f from $\mathbb{B}^n$ to $\mathbb{B}^N$, we consider a proper holomorphic map from $\mathbb{B}^n$ to $\mathbb{B}^{N+k}$ defined by $z \mapsto (f(z), 0, \ldots, 0)$ with k-zeros for which we will use the same notation f if there is no confusion. When $n \geq 3$ and $N \leq 2n - 2$, Faran ([9]) showed that it is equivalent to f_s if it is extended holomorphically over the boundary. Furthermore he precisely classified the equivalence classes of proper holomorphic maps from $\mathbb{B}^2$ to $\mathbb{B}^3$ which is C^3-smooth up to the boundary ([8]). In [10], Forstnerič proved that any proper holomorphic map from $\mathbb{B}^n$ to $\mathbb{B}^N$ which is C^{N-n+1}-smooth up to the boundary is a rational map $(p_1, \ldots, p_N)/q$ where p_j and q are holomorphic polynomials of degree at most $N^2(N - n + 1)$. Since this work has been done, much results fit into the framework of providing sharp bounds in special situations. See [5, 7, 16], for more details. If $N = 2n - 1$, there is a proper holomorphic map which is called the Whitney map $f_w : \mathbb{B}^n \to \mathbb{B}^{2n-1}$ defined by

$$f_w(z_1, \ldots, z_n) = (z_1, \ldots, z_{n-1}, z_n z_1, z_n z_2, \ldots, z_n^2).$$

It is inequivalent to the embedding f_s. Moreover Huang and Ji ([13]) proved that any proper rational map from $\mathbb{B}^n$ to $\mathbb{B}^{2n-1}$ with $n \geq 3$ is equivalent to f_s or f_w and any proper holomorphic embedding which is C^2-smooth up to the boundary is equivalent to f_s. If the dimension of the target domain is larger than $2n$, there are infinitely many inequivalent proper holomorphic maps. For example, $f_\theta : \mathbb{B}^n \to \mathbb{B}^{2n}$ given by

$$(1.1) \qquad f_\theta(z) = (z_1, \ldots, z_{n-1}, \cos\theta z_n, \sin\theta z_1 z_n, \ldots, \sin\theta z_n z_n)$$

with $0 \leq \theta \leq \frac{\pi}{2}$ are found by D'Angelo ([3]). In [3], D'Angelo showed that any two proper holomorphic polynomial map from $\mathbb{B}^n$ to $\mathbb{B}^N$ preserving the origin are equivalent if and only if they are isotropically equivalent at the origin and as a consequence (1.1) are inequivalent for all $0 \leq \theta \leq \frac{\pi}{2}$. Interestingly, Hamada ([11]) showed that any proper rational map from $\mathbb{B}^n$ to $\mathbb{B}^{2n}$ with $n \geq 4$ is equivalent to f_θ for some θ, $0 \leq \theta \leq \pi/2$ and Huang, Ji and Xu ([15]) showed that any proper holomorphic map from $\mathbb{B}^n$ to $\mathbb{B}^N$ with $4 \leq n \leq N \leq 3n - 4$ which is C^3-smooth up to the boundary should be equivalent to f_θ for some θ, $0 \leq \theta \leq \pi/2$. Recently Huang-Ji-Yin ([14]) proved that any proper rational map from $\mathbb{B}^n$ to $\mathbb{B}^N$ with $n \geq 8$ and $3n + 1 \leq N \leq 4n - 7$ should be equivalent to proper rational map from $\mathbb{B}^n$ to $\mathbb{B}^{3n}$.

As a one generalization of the unit ball, one consider bounded symmetric domains which are Hermitian symmetric spaces of non-compact type with non-smooth boundaries. There are several rigidity theorems on proper holomorphic maps between bounded symmetric domains of rank greater than or equal two. In contrast

with the case of the unit balls, the difference of the ranks between the domains is more crucial than that of the dimensions. The first result on bounded symmetric domains along these lines is the following which is due to Tsai. Let $f : \Omega_1 \to \Omega_2$ be a proper holomorphic map between irreducible bounded symmetric domains Ω_1 and Ω_2. If $\text{rank}(\Omega_1) \geq \text{rank}(\Omega_2)$, then $\text{rank}(\Omega_1) = \text{rank}(\Omega_2)$ and f should be a totally geodesic isometric embedding with respect to the Bergman metrics on the domains (see [20]). If $\Omega_1 = \Omega_{r,r-1}^I$ and $\Omega_2 = \Omega_{r,r}^I$, then f is also a totally geodesic isometric embedding (see [21]), although the rank of Ω_2 is larger than that of Ω_1. Furthermore, Ng ([18]) showed that for $f : \Omega_{r,s}^I \to \Omega_{r',s'}^I$, if $s \geq 2$, $s \geq r' \geq r$ and $r' \leq 2r - 1$, then f is equivalent to the embedding, $Z \mapsto \begin{pmatrix} Z & 0 \\ 0 & 0 \end{pmatrix}$. If the difference of the ranks of the domains gets bigger, then it is expected that there are lots of inequivalent proper holomorphic maps. In [19], one way of finding proper holomorphic maps between bounded symmetric domains of type I is suggested and several proper holomorphic maps are constructed. For example, for $r' = 2r - 1$ and $s' = 2s - 1$, there is a generalized Whitney map defined by

$$
(1.2) \quad \begin{pmatrix} z_{11} & \cdots & z_{1s} \\ \vdots & \ddots & \vdots \\ z_{r1} & \cdots & z_{rs} \end{pmatrix} \mapsto \begin{pmatrix} z_{11}^2 & z_{11}z_{12} & \cdots & z_{11}z_{1s} & z_{12} & \cdots & z_{1s} \\ z_{11}z_{21} & z_{21}z_{12} & \cdots & z_{21}z_{1s} & z_{22} & \cdots & z_{2s} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ z_{11}z_{r1} & z_{r1}z_{12} & \cdots & z_{r1}z_{1s} & z_{r2} & \cdots & z_{rs} \\ z_{21} & z_{22} & \cdots & z_{2s} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ z_{r1} & z_{r2} & \cdots & z_{rs} & 0 & \cdots & 0 \end{pmatrix}.
$$

In this paper, as a one step to observe analogous phenomenon on proper holomorphic maps between bounded symmetric domains of rank greater than or equal to two, we generalize the result of D'Angelo in [3] to the domains of classical type.

2. Preliminaries

In this section, we introduce terminology and some basic background. A bounded domain Ω is called *symmetric* if for each $p \in \Omega$, there is a holomorphic automorphism $\mathfrak{i}_p$ such that $\mathfrak{i}_p^2$ is the identity map of Ω which has p as an isolated fixed point. All bounded symmetric domains are homogeneous domain, i.e. the automorphism group acts transitively on the domain. In 1920's, Cartan classified all irreducible bounded symmetric domains. There are 4 classical types and 2 exceptional types. The four classical types are given by (1),(2),(3) and (4) in the introduction. Note that $\Omega_{m,1}^I$ is the m-dimensional unit ball and Ω_1^{III} is the unit disc.

From now on, we will use the notation $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL(r + s, \mathbb{C})$ to split M into 4 block matrices with $A \in M_{r,r}^{\mathbb{C}}$, $B \in M_{r,s}^{\mathbb{C}}$, $C \in M_{s,r}^{\mathbb{C}}$ and $D \in M_{s,s}^{\mathbb{C}}$. We will denote by $ASM_{n,n}^{\mathbb{C}}$ the set of anti-symmetric complex $n \times n$ matrices and by $SM_{n,n}^{\mathbb{C}}$ the set of symmetric complex $n \times n$ matrices.

Let $U(r,s)$ be the subgroup of $GL(r+s,\mathbb{C})$ satisfying

$$M \begin{pmatrix} -I_r & 0 \\ 0 & I_s \end{pmatrix} M^* = \begin{pmatrix} -I_r & 0 \\ 0 & I_s \end{pmatrix}$$

for all $M \in U(r,s)$. Let $SU(r,s)$ be the subset of $U(r,s)$ which consists of the matrices with determinant one. Explicitly,

$$SU(r,s) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL(r+s,\mathbb{C}) : AA^* - BB^* = I_r, \right.$$

(2.1)

$$\left. AC^* = BD^*, \ CC^* - DD^* = -I_s \right\}.$$

Let $O(n+2,\mathbb{C})$ be the complex orthogonal group of $(n+2) \times (n+2)$ matrices.

Since every bounded symmetric domain is Hermitian symmetric space of non-compact type, the domain can be canonically embedded into the corresponding compact dual (Borel embedding) and every holomorphic automorphism of the domain can be extended to the automorphism of its compact dual. For example, let $G_{r,s}$ be the Grassmannian of r-planes in $r+s$ dimensional complex vector space $\mathbb{C}^{r+s}$ which is the compact dual of $\Omega_{r,s}^I$. For $X \in M_{r,r+s}^{\mathbb{C}}$ of rank r, denote $[X]$ the r-plane in $\mathbb{C}^{r+s}$ which is generated by row vectors of X. For $M \in GL(r+s,\mathbb{C})$, M acts on $G_{r,s}$ by $[X] \in G_{r,s} \mapsto [XM]$. Then the Borel embedding ξ^I of $\Omega_{r,s}^I$ is given by

$$\xi^I(Z) = [I_r, Z] \in G_{r,s}$$

and $M \in U(r,s)$ acts on $\Omega_{r,s}^I$ by $g_M(Z) := \xi^{I^{-1}}(\xi(Z)M)$. Explicitly, for $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U(r,s)$, M acts on $\Omega_{r,s}^I$ by

(2.2)
$$Z \mapsto (A+ZC)^{-1}(B+ZD).$$

Similarly, for $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U(n,n)$ acts on Ω_n^{II} and Ω_n^{III} by (2.2).

In case of Ω_n^{IV}, the explicit expression of the holomorphic automorphism is little more messy. The compact dual of Ω_n^{IV} is the hyperquadric H_n in $\mathbb{P}^{n+1}$ given by $H_n := \{[z_1,\ldots,z_{n+2}] \in \mathbb{P}^{n+1} : z_1^2 + \cdots + z_n^2 - z_{n+1}^2 - z_{n+2}^2 = 0\}$. Then the Borel embedding ξ^{IV} is given by

(2.3)
$$\xi^{IV}(Z) = [-2iZ, 1+ZZ^t, i(1-ZZ^t)] \in H_n$$

For $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in O(n+2,\mathbb{C}) \cap U(n,2)$, then M acts on Ω_n^{IV} by

$$Z = (z_1,\ldots,z_n) \mapsto \frac{1}{(-2iZB + Z'D)(i,1)^t}(2iZA - Z'C)$$

where $Z' = (1+ZZ^t, \ i-iZZ^t)$.

The automorphism groups of classical domains and their isotropy groups at the origin are given by the following:

(1) Aut $\left(\Omega_{r,s}^I\right) = U(r,s)$, Isot $\left(\Omega_{r,s}^I\right) = \left\{ \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} : U \in U(r), \ V \in U(s) \right\}$

(2) $\text{Aut}\left(\Omega_n^{II}\right) = \left\{ M \in U(n,n) : M^t \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} M = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \right\}$,

$\text{Isot}\left(\Omega_n^{II}\right) = \left\{ \begin{pmatrix} A & 0 \\ 0 & \overline{A} \end{pmatrix} : A \in U(n) \right\}$

(3) $\text{Aut}\left(\Omega_n^{III}\right) = \left\{ M \in U(n,n) : M^t \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} M = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \right\}$,

$\text{Isot}\left(\Omega_n^{III}\right) = \left\{ \begin{pmatrix} A & 0 \\ 0 & \overline{A} \end{pmatrix} : A \in U(n) \right\}$

(4) $\text{Aut}(\Omega_n^{IV}) = O(n+2,\mathbb{C}) \cap U(n,2), \quad \text{Isot}(\Omega^{IV}) = O(n) \times O(2)$

3. ISOTROPICALLY EQUIVALENT PROPER HOLOMORPHIC POLYNOMIAL MAPS

In this section, we will prove Theorem 1.1. Define a polynomial function $S_{r,s}^I : \Omega_{r,s}^I \to \mathbb{R}$ by

$$S_{r,s}^I(Z) = \det(I_r - ZZ^*)$$

for $Z = (z_{ij}) \in M_{r,s}^{\mathbb{C}}$, as a real polynomial in $\text{Re}\,(z_{ij})$, $\text{Im}\,(z_{ij})$ where $1 \leq i \leq r$, $1 \leq j \leq s$. $S_{r,s}^I$ is a polynomial of degree 2 in each $\text{Re}\,(z_{ij})$, $\text{Im}\,(z_{ij})$. In case of Ω_n^{II}, it is known that $\det(I_r - ZZ^*) = s_n^{II}(Z)^2$ for some polynomial $s_n^{II}(Z)$ (cf.[17]). Define $S_n^{II} : \Omega_n^{II} \to \mathbb{R}$ and $S_n^{III} : \Omega_n^{III} \to \mathbb{R}$ by

$$S_n^{II}(Z) = s_n^{II}(Z) \ \text{ for } Z \in ASM_{n,n}^{\mathbb{C}},$$

$$S_n^{III}(Z) = \det(I_n - ZZ^*) \ \text{ for } Z \in SM_{n,n}^{\mathbb{C}}$$

and

$$S_n^{IV}(Z) = 1 - 2ZZ^* + \left|ZZ^t\right|^2 \ \text{ for } Z \in \mathbb{C}^n$$

$S_{r,s}^I(Z)$, $S_n^{II}(Z)$, $S_n^{III}(Z)$ and $S_n^{IV}(Z)$ are called the *generic norm* of the corresponding domains cf.[17]. Then $S_n^{II}(Z)$ is a polynomial of degree 2 in each $\text{Re}\,z_{ij}, \text{Im}\,z_{ij}$ for $1 \leq i < j \leq n$ and $S_n^{III}(Z)$ is a polynomial of degree 4 in each $\text{Re}\,z_{ij}, \text{Im}\,z_{ij}$ for $1 \leq i < j \leq n$ and of degree 2 in each $\text{Re}\,z_{ii}, \text{Im}\,z_{ii}$ for $1 \leq i \leq n$.

Lemma 3.1.

(1) *For $Z = (z_{ij}) \in M_{r,s}^{\mathbb{C}}$, the coefficient of $(\text{Re}\,z_{ij})^2$ in $S_{r,s}^I(Z)$ is*

$$-\det(I_{r-1} - Z'Z'^*)$$

where Z' is the (i,j) minor of Z.

(2) *For $Z = (z_{ij}) \in ASM_{n,n}^{\mathbb{C}}$, the coefficient of $(\text{Re}\,z_{ij})^4, 1 \leq i < j \leq n$ in $S_n^{II}(Z)$ is*

$$\det(I_{n-2} - Z''Z''^*)$$

where Z'' is $(n-2) \times (n-2)$ matrix obtained by removing i,j-th rows and i,j-th columns in Z.

(3) *For $Z = (z_{ij}) \in SM_{n,n}^{\mathbb{C}}$, the coefficient of $(\text{Re}\,z_{ij})^4, 1 \leq i < j \leq n$ and $(\text{Re}\,z_{ij})^2$ for $1 \leq i \leq n$ in $S_n^{III}(Z)$ are*

$$\det(I_{n-2} - Z''Z''^*) \quad and \quad -\det(I_{n-1} - Z'Z'^*)$$

respectively where Z'' is $(n-2) \times (n-2)$ matrix removing i,j-th rows and i,j-th columns in Z and Z' is the (i,i) minor of Z.

Proof.

(1). For $Z \in M_{r,s}^{\mathbb{C}}$, denote $Z = \begin{pmatrix} X \\ Y \end{pmatrix}$ with $X = (X', X'') = (x_{ij}) \in M_{r-1,s}^{\mathbb{C}}$ where $X' \in M_{r-1,1}^{\mathbb{C}}$, $X'' \in M_{r-1,s-1}^{\mathbb{C}}$ and $Y = (y_1, \ldots, y_s) \in M_{1,s}^{\mathbb{C}}$. We only consider the coefficient of $(\operatorname{Re} y_1)^2$. The coefficient of $(\operatorname{Re} y_1)^2$ in $S_{r,s}^I(Z)$ is $\frac{\partial^2}{\partial \overline{y}_1 \partial y_1} S_{r,s}^I(Z)$. Since

$$(3.1) \qquad \frac{\partial}{\partial y_1} \det(I_r - ZZ^*) = \frac{\partial}{\partial y_1} \det \begin{pmatrix} I_{r-1} - XX^* & -XY^* \\ -YX^* & 1 - YY^* \end{pmatrix}$$

and $YX^* = (y_1 \overline{x}_{11} + \cdots + y_s \overline{x}_{1s}, \ldots, y_1 \overline{x}_{(r-1)1} + \cdots + y_s \overline{x}_{(r-1)s})$, we obtain

$$\frac{\partial}{\partial y_1} \det(I_r - ZZ^*) = \det \begin{pmatrix} I_{r-1} - XX^* & -XY^* \\ (-\overline{x}_{11}, -\overline{x}_{21}, \ldots, -\overline{x}_{(r-1)1}) & -\overline{y}_1 \end{pmatrix}$$

and

$$(3.2) \qquad \frac{\partial^2}{\partial \overline{y}_1 \partial y_1} \det(I_r - ZZ^*)$$

$$(3.3) \qquad = \det \left(\begin{array}{c|c} I_{r-1} - XX^* & \begin{pmatrix} -x_{11} \\ \vdots \\ -x_{(r-1)1} \end{pmatrix} \\ \hline (-\overline{x}_{11}, -\overline{x}_{21}, \ldots, -\overline{x}_{(r-1)1}) & -1 \end{array} \right)$$

$$= \det \left(\begin{array}{c|c} I_{r-1} - X''X''^* & 0 \\ \hline (-\overline{x}_{11}, -\overline{x}_{21}, \ldots, -\overline{x}_{(r-1)1}) & -1 \end{array} \right)$$

$$(3.4) \qquad = -\det(I_{r-1} - X''X''^*).$$

The second equation comes from subtracting j-th row of (3.3) by x_{j1} times the r-th row of (3.3).

(2). We will only consider the coefficient of $(\operatorname{Re} z_{12})^4$. By Lemma 3.1(1), for $W = (w_{ij}) \in M_{n,n}^{\mathbb{C}}$, $\det(I_n - WW^*) = a_2 (\operatorname{Re} w_{12})^2 + a_1 (\operatorname{Re} w_{12}) + a_0$ where a_i are polynomials in $\operatorname{Re} w_{ij}, \operatorname{Im} w_{ij}$ for $i \neq 1$, $j \neq 2$ and $\operatorname{Im} w_{12}$. Since the coefficient of $(\operatorname{Re} z_{12})^4$ is the coefficient of $(\operatorname{Re} w_{12})^2 (\operatorname{Re} w_{21})^2$ substituted $w_{ij} = z_{ij}$ for $1 \leq i < j < n$, $w_{ji} = -z_{ij}$ for $1 \leq i < j \leq n$ and $w_{ii} = 0$ for $1 \leq i \leq n$, the coefficient of $(\operatorname{Re} z_{12})^4$ is $\det(I_{n-2} - Z''Z''^*)$ where $Z'' = (z_{ij})_{3 \leq i \leq n, \, 3 \leq j \leq n}$.

(3). We can obtain the result by similar method in (1) and (2). $\qquad \square$

Proposition 3.2. $S_{r,s}^I(Z)$, $S_n^{II}(Z)$, $S_n^{III}(Z)$ and $S_n^{IV}(Z)$ are irreducible.

Proof. **In case of $S_{r,s}^I$:** At first, we will prove that $S_{r,s}^I(Z)$ is irreducible. Note that $r \leq s$. We use induction with respect to k on $S_{k,s-r+k}^I$. For $Z \in M_{1,s-r+1}^{\mathbb{C}}$, $1 - ZZ^*$ is irreducible. Suppose that $S_{r-1,s-1}^I(Z)$ is irreducible and $S_{r,s}^I(Z) = A B$ for some polynomial A and B. Denote $Z = \begin{pmatrix} X \\ Y \end{pmatrix}$ as in the proof of Lemma 3.1. The degree of $\operatorname{Re} y_1$ is 2.

Step 1 : Suppose that there is a nonzero monomial of $(\operatorname{Re} y_1)^2$ in A. Then $S_{r,s}^I(Z)$ should be of the form

$$S_{r,s}^I(Z) = (\mu(\operatorname{Re} y_1)^2 + \sigma \operatorname{Re} y_1 + \nu) B$$

where μ, σ, ν and B are polynomials without $\operatorname{Re} y_1$ variable. Without loss of generality, we may assume that B is not a constant. Then by Lemma 3.1, μB is $\det(I_{r-1} - X''X''^*)$ which is irreducible by the induction hypothesis. Hence B

should be $\det(I_{r-1} - X''X''^*)$ (up to constants). This implies that it consists of the monomials of variable X''. Then every variable of X' should be in A. Consider the coefficient of $(\operatorname{Re} x_{11})^2$ which is irreducible and does not contain $\operatorname{Re} x_{1j}$, $\operatorname{Im} x_{1j}$ variables. However the coefficient of $(\operatorname{Re} x_{11})^2$ should contain B and this induces a contradiction. Therefore there is no second order term in each A and B.

Step 2 : Suppose that $S_{r,s}^I(Z) = (\mu \operatorname{Re} y_1 + \sigma)(\nu \operatorname{Re} y_1 + \rho)$ where μ, σ, ν, ρ has no $\operatorname{Re} y_1$ variable. Then $\mu\nu$ is irreducible by Lemma 3.1 and hence μ or ν is a constant. If μ is a constant, $\nu = \det(I_{r-1} - X''X''^*)$ up to constant. Note that ν contains second order terms of variables in X''. Since there is no $\operatorname{Re} y_1$ term in ρ, $\nu \operatorname{Re} y_1 + \rho$ should have second order term. But by Step 1, $\nu \operatorname{Re} y_1 + \rho$ cannot have second order term. Hence $S_{r,s}^I(Z)$ is irreducible.

In case of S_n^{II}, S_n^{III}, S_n^{IV}: Since for $n = 1$, $S_1^{III}(z) = 1 - |z|^2$ for $z \in \Delta$ which is irreducible, use the same method (induction) as in the proof of the case $S_{r,s}^I$ considering factorization with respect to $\operatorname{Re} z_{11}$ variable. In case of S_n^{IV}, we can easily show that it is irreducible. We omit the proof.

We only need to prove that $S_n^{II}(Z)$ is irreducible. For $n = 2$, let $Z = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$. Then $S_2^{II}(Z) = 1 - |a|^2$ which is irreducible. For $n = 3$, let $Z = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$. Then $S_3^{II}(Z) = 1 - |a|^2 - |b|^2 - |c|^2$ which is also irreducible. Assume that $S_{2n}^{II}(Z)$ and $S_{2n-1}^{II}(Z)$ are irreducible. Since the even dimensional case is similar to the odd dimensional case, we will only consider the odd dimensional case. Since the coefficient of $(\operatorname{Re} z_{ij})^4$ in $S_{2n+1}^{II}(Z)^2$ is $\det(I_{2n-1} - Z''Z''^*)$ as in Lemma 3.1 (2), the coefficient of $(\operatorname{Re} z_{ij})^2$ in $S_{2n+1}^{II}(Z)$ is irreducible. Hence similar proof of the case $S_{r,s}^I$ can be applied. $\square$

Let $f : \Omega_1 \to \Omega_2$ be a proper holomorphic polynomial map where Ω_1, Ω_2 are irreducible bounded symmetric domains of classical type. Let S_1, S_2 be the corresponding generic norms. Since f is proper, by Proposition 3.2,

$$S_2(f(Z)) = 0 \text{ whenever } S_1(Z) = 0.$$

(Note that if $Z \in \partial\Omega_n^{IV}$, $S_n^{IV}(Z) = 0$ since if $ZZ^* = 1$, $S_n^{IV}(Z) < 0$. Hence we do not need to consider $ZZ^* - 1$ term in the definition of Ω_n^{IV}.) Hence there is a real analytic map F_f such that

$$(3.5) \qquad\qquad S_2(f(Z)) = S_1(Z)\, F_f(Z).$$

We can polarize this equation by

$$(3.6) \qquad\qquad S_2(f(Z), f(W)) = S_1(Z, W)\, F_f(Z, W).$$

Example 3.3.   (1) Let $f : \Omega_{r,s} \to \Omega_{r',s'}$ be a proper holomorphic polynomial map. Then (3.6) is expressed by

$$\det(I_{r'} - f(Z)f(W)^*) = \det(I_r - ZW^*)\, F_f(Z, W).$$

(2) Let $f : \Omega_n^{IV} \to \Omega_N^{IV}$ be a proper holomorphic polynomial map. Then (3.6) is expressed by

$$1 - 2f(Z)f(W)^* + \left(f(Z)f(Z)^t\right)\overline{\left(f(W)f(W)^t\right)}$$
$$= \left(1 - 2ZW^* + (ZZ^t)(\overline{WW^t})\right) F_f(Z, W).$$

Lemma 3.4. *Let $g : \Omega_1 \to \Omega_2$ and $f : \Omega_2 \to \Omega_3$ be proper holomorphic polynomial maps. Then*

$$F_{f \circ g}(Z, W) = F_g(Z, W) \, F_f(g(Z), g(W)).$$

Proof.

$$
\begin{aligned}
S_1(Z, W) F_{f \circ g}(Z, W) &= S_3(f \circ g(Z), f \circ g(W)) \\
&= S_2(g(Z), g(W)) \, F_f(g(Z), g(W)) \\
&= S_1(Z, W) F_g(Z, W) \, F_f(g(Z), g(W))
\end{aligned}
$$

$\square$

Lemma 3.5. *Let $U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be an automorphism of a domain Ω with appropriate block matrices A, B, C, D. Then F_U is given as the following:*

(1) *If $\Omega = \Omega_{r,s}^I$ or Ω_n^{III},*

$$F_U(Z, W) = \frac{1}{\det(A + ZC)\overline{\det(A + WC)}}.$$

(2) *If $\Omega = \Omega_n^{II}$,*

$$F_U(Z, W) = \frac{1}{f_U(z, w)} \quad \text{where} \quad f_U(Z, W)^2 = \det(A + ZC)\overline{\det(A + WC)}.$$

(3) *If $\Omega = \Omega_n^{IV}$,*

$$F_U(Z, W) = -\frac{1}{2} \frac{1}{\{(-2iZB + Z'D)(i,1)^t\}\overline{\{(-2iWB + W'D)(i,1)^t\}}}$$

Proof. (1 and 2) Since by (2.1),

$$
\begin{aligned}
U(Z)U(W)^* &= (A + ZC)^{-1}(B + ZD)\{(A + WC)^{-1}(B + WD)\}^* \\
&= (A + ZC)^{-1}\left((A + ZC)(A + WC)^* + ZZ^* - I_r\right)\left((A + WC)^{-1}\right)^* \\
&= I_r - (A + ZC)^{-1}(I_r - ZZ^*)\left((A + WC)^{-1}\right)^*,
\end{aligned}
$$

$$(3.7) \qquad \det(I_r - U(Z)U(W)^*) = \frac{\det(I_r - ZW^*)}{\det(A + ZC)\overline{\det(A + WC)}}.$$

(3) Note that for $Z = (z_1, \ldots, z_n)$, $S_n^{IV}(Z) = -\frac{1}{2}Q(\tilde{\xi}^{IV}(Z))$ where $Q(x_1, \ldots, x_{n+2}) = |x_1|^2 + \ldots |x_n|^2 - |x_{n+1}|^2 - |x_{n+2}|^2$ and $\tilde{\xi}^{IV}(Z) = (-2iZ, 1 + ZZ^t, i(1 - ZZ^t))$. Then

$$
\begin{aligned}
S_n^{IV}(g_M(Z)) &= -\frac{1}{2}Q\left(\tilde{\xi}^{IV}(\xi^{IV^{-1}}(\xi^{IV}(Z)M))\right) \\
&= -\frac{1}{2}Q\left(\frac{-2iZA + Z'C}{(-2iZB + Z'D)(i,1)^t}, \frac{-2iZB + Z'D}{(-2iZB + Z'D)(i,1)^t}\right) \\
&= -\frac{1}{2}\frac{S_n^{IV}(Z)}{|(-2iZB + Z'D)(i,1)^t|^2}
\end{aligned}
$$

$\square$

Remark 3.6. In case of Ω_n^{II}, since for $Z \in ASM_{n,n}^{\mathbb{C}}$, $C^t A + C^t ZC$ is anti-symmetric. This implies that

$$\det(A + ZC) = \frac{\det(C^t A + C^t ZC)}{\det(C)} = \frac{(Pf(C^t A + C^t ZC))^2}{\det(C)}$$

where $Pf(Y)$ is the Pfaffian polynomial of a matrix Y and hence F_U is a rational function.

Proof of Theorem 1.1. Suppose that $g \circ U = V \circ f$ for some $V = \begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix} \in$ Aut(Ω_2) and $U = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix} \in$ Aut(Ω_1). Then since $F_{g \circ U} = F_{V \circ f}$, by Lemma 3.4,

$$F_g(U(Z), U(W)) F_U(Z, W) = F_V(f(Z), f(W)) F_f(Z, W).$$

By multiplying $S^1(Z, W)$ to both side, we obtain

(3.8) $\quad S^2(g \circ U(Z), g \circ U(W)) = F_V(f(Z), f(W)) S^1(f(Z), f(W)).$

For simplicity, we only consider that the case Ω_1 and Ω_2 are bounded symmetric domains of type I. (3.8) is

$$\det(I_{r'} - f(Z) f(W)^*) = \det(I_{r'} - g \circ U(Z)(g \circ U(W))^*)$$

(3.9)
$$\det(V_1 + f(Z) V_3) \overline{\det(V_1 + f(W) V_3)}.$$

Put $W = 0$ in (3.9). Then

(3.10) $\quad 1 = \det(I_{r'} - g \circ U(Z)(g \circ U(0))^*) \det(V_1 + f(Z) V_3) \overline{\det(V_1)}.$

If $U_3 = 0$, then by $g(U_1^{-1} Z U_4) = V \circ f(Z)$, we obtain $0 = g(0) = V \circ f(0) = V_1^{-1} V_2$ and hence $V_2 = 0$. So assume $U_3 \neq 0$. Note that in this case, $\det(U_1 + Z U_3)$ is not a constant. Suppose that $g \circ U(0) \neq 0$. Then $\det(I_{r'} - g \circ U(Z)(g \circ U(0))^*)$ is not a constant and hence it should be of the form p/q where p and q are non-constant polynomial without common factors and $q = \det(U_1 + Z U_3)^l$ for some l. But since product of p/q and polynomial cannot be a constant, (3.10) induces a contradiction. Hence $g \circ U(0)$ should be zero. This implies that $0 = g(U_1^{-1} U_2) = V \circ f(0) = V(0) = V_1^{-1} V_2$. Hence $V_2 = 0$ (and also $V_3 = 0$) and hence $1 = \det(V_1 + f(Z) V_3)$. Put this in (3.9).

(3.11) $\quad \det(I_{r'} - f(Z) f(W)^*) = \det(I_{r'} - g \circ U(Z)(g \circ U(W))^*).$

Since right side of (3.11) is singular on $\{Z \in \Omega_{r,s} : \det(U_1 + Z U_3) = 0\}$, U_3 should be zero. $\qquad\square$

4. APPLICATION

In this section, we suggest examples which are 1-parameter family of inequivalent proper holomorphic maps between bounded symmetric domains of classical type. We use Theorem 1.1 to prove that proper holomorphic maps $f_t : \Omega_1 \rightarrow \Omega_2$ are inequivalent for each $0 \leq t \leq 1$. Every example in this section is obtained in [19]. As in [6], we define the following equivalence relation:
Let Ω_1, Ω_2 be domains. Consider a continuous map $H : \Omega_1 \times [0, 1] \rightarrow \Omega_2$. Denote $H_t(z) = H(z, t)$. Suppose that $H_t : \Omega_1 \rightarrow \Omega_2$ is holomorphic for each $t \in [0, 1]$. Then we will say H_t is a *continuous family* of holomorphic maps from Ω_1 to Ω_2.

Definition 4.1. Let $f : \Omega_1 \rightarrow \Omega_2$ and $g : \Omega_1 \rightarrow \Omega_3$ be proper holomorphic maps. Then f and g are *homotopic in the target domain* Ω if for each $t \in [0, 1]$, there is a proper holomorphic maps $H_t : \Omega_1 \rightarrow \Omega$ such that

- there are totally geodesic embeddings $e_k : \Omega_k \rightarrow \Omega$ for $k = 2, 3$, with respect to their Bergman metrics,
- $H_0 = e_2 \circ f$ and $H_1 = e_3 \circ g$,

- H_t is a continuous family of holomorphic maps from Ω_1 to Ω.

4.1. **1-parameter family of proper holomorphic maps among $\Omega_{r,s}^I$.**
Consider proper holomorphic maps $f, g : \Omega_{2,2}^I \to \Omega_{3,3}^I$ which are defined by

$$f\left(\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}\right) = \begin{pmatrix} z_1^2 & z_1 z_2 & z_2 \\ z_1 z_3 & z_2 z_3 & z_4 \\ z_3 & z_4 & 0 \end{pmatrix}, \text{ for } \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \in \Omega_{2,2}^I$$

$$g\left(\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}\right) = \begin{pmatrix} z_1^2 & \sqrt{2}z_1 z_2 & z_2^2 \\ \sqrt{2}z_1 z_3 & z_1 z_4 + z_2 z_3 & \sqrt{2}z_2 z_4 \\ z_3^2 & \sqrt{2}z_3 z_4 & z_4^2 \end{pmatrix}$$

and $f_t : \Omega_{2,2}^I \to \Omega_{4,4}^I$ be proper holomorphic maps for $0 \le t \le 1$ defined by
(4.1)

$$\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \mapsto \begin{pmatrix} z_1^2 & \sqrt{2-t}z_1 z_2 & \sqrt{1-t}z_2^2 & \sqrt{t}z_2 \\ \sqrt{2-t}z_1 z_3 & \frac{2(1-t)}{2-t}z_1 z_4 + z_2 z_3 & 2\sqrt{\frac{1-t}{2-t}}z_2 z_4 & \sqrt{\frac{t}{2-t}}z_4 \\ \sqrt{1-t}z_3^2 & 2\sqrt{\frac{1-t}{2-t}}z_3 z_4 & z_4^2 & 0 \\ \sqrt{t}z_3 & \sqrt{\frac{t}{2-t}}z_4 & 0 & 0 \end{pmatrix}.$$

Then it is easily observed that f and g are homotopic in the target domain $\Omega_{4,4}^I$ through f_t.

Corollary 4.2. *f_t are inequivalent for different t, $0 \le t \le 1$.*

Proof. Suppose that $f_t \circ A = B \circ f_s$ for some $A \in U(2,2)$ and $B \in U(4,4)$. Without loss of generality, we may assume that $t \ne 0$. Then by Theorem 1.1, $f_t(UZV) = Lf_s(Z)M$ for some $U = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix} \in U(2)$, $V = \begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix} \in U(2)$ and $L = (L_{ij})$, $M = (M_{ij}) \in U(4)$. Denote $f_t = \sum f_{t,j}$ where $f_{t,j}$ is a homogeneous polynomial of degree j. Then $f_{t,j}(UZV) = Lf_{s,j}(Z)M$ for each j. Consider linear part

$$f_{t,1} = \begin{pmatrix} 0 & 0 & 0 & \sqrt{t}z_2 \\ 0 & 0 & 0 & \sqrt{\frac{t}{2-t}}z_4 \\ 0 & 0 & 0 & 0 \\ \sqrt{t}z_3 & \sqrt{\frac{t}{2-t}}z_4 & 0 & 0 \end{pmatrix}.$$

At first, at $Z_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$,

$$f_{t,1}(UZ_1V) = f_{t,1}\begin{pmatrix} U_1V_1 & U_1V_2 \\ U_3V_1 & U_3V_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \sqrt{t}U_1V_2 \\ 0 & 0 & 0 & \sqrt{\frac{t}{2-t}}U_3V_2 \\ 0 & 0 & 0 & 0 \\ \sqrt{t}U_3V_1 & \sqrt{\frac{t}{2-t}}U_3V_2 & 0 & 0 \end{pmatrix}$$

and $Lf_{s,1}(Z_1)M = 0$. This implies that $U_1V_2 = U_3V_2 = U_3V_1 = U_3V_2 = 0$. Suppose that $U_3 \ne 0$. Then $V_1 = V_2 = 0$. This is a contradiction to $V \in U(2)$. Hence $U_3 = 0$. This implies that $U_2 = 0$ and $U_1 \ne 0$, hence $V_2 = V_3 = 0$.

Second, at $Z_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$,

$$f_{t,1}(UZ_2V) = f_{t,1}\begin{pmatrix} 0 & U_1V_4 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \sqrt{t}U_1V_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$Lf_{s,1}(Z_2)M = \sqrt{s}\begin{pmatrix} L_{11}(M_{41},\, M_{42},\, M_{43},\, M_{44}) \\ L_{21}(M_{41},\, M_{42},\, M_{43},\, M_{44}) \\ L_{31}(M_{41},\, M_{42},\, M_{43},\, M_{44}) \\ L_{41}(M_{41},\, M_{42},\, M_{43},\, M_{44}) \end{pmatrix}.$$

This implies that $L_{21} = L_{31} = L_{41} = 0$, $L_{11} \neq 0$, $M_{41} = M_{42} = M_{43} = 0$ and $\sqrt{t}U_1V_4 = \sqrt{s}L_{11}M_{44}$. Since L is unitary, $L_{12} = L_{13} = L_{14} = 0$. Hence by taking square norm on $\sqrt{t}U_1V_4 = \sqrt{s}L_{11}M_{44}$, we obtain $t = s$. $\square$

Remark 4.3. In [19], there are generalizations of f_0, f_1 to $\tilde{f}_0$, $\tilde{f}_1$ as proper holomorphic maps $\tilde{f}_0 : \Omega^I_{r,s} \to \Omega^I_{2r-1,2s-1}$ and $\tilde{f}_1 : \Omega^I_{r,s} \to \Omega^I_{\frac{1}{2}r(r+1),\frac{1}{2}s(s+1)}$. Similar to (4.1), we can make homotopy $\tilde{f}_t$ of $\tilde{f}_0$ and $\tilde{f}_1$ which are inequivalent for all t, $0 \leq t \leq 1$. That is, $\tilde{f}_0$ and $\tilde{f}_1$ are homotopic in the target domain $\Omega^I_{r',s'}$ with $r' = \frac{1}{2}(r^2 + r + 2s - 2)$ and $s' = \frac{1}{2}(s^2 + s + 2r - 2)$.

There are only two inequivalent proper holomorphic maps from $\mathbb{B}^n$ to $\mathbb{B}^{2n-1}$ which are the standard embedding and the Whitney map (see [13]). As the case of the unit balls, it is expected that there are only two inequivalent proper holomorphic maps from $\Omega^I_{r,s}$ to $\Omega^I_{2r-1,2s-1}$ that are the standard embedding and the generalized Whitney map (1.2). In contrast to $\Omega^I_{2r-1,2s-1}$, if the target domain changes to $\Omega_{2r-1,2s}$, there are infinitely many proper holomorphic maps; Let $g_t : \Omega^I_{2,2} \to \Omega^I_{3,4}$, $0 \leq t \leq 1$ be proper holomorphic maps defined by

(4.2) $$\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \mapsto \begin{pmatrix} \sqrt{t}z_1^2 & \sqrt{t}z_1z_2 & \sqrt{1-t}z_1 & z_2 \\ \sqrt{t}z_1z_3 & \sqrt{t}z_2z_3 & \sqrt{1-t}z_3 & z_4 \\ z_3 & z_4 & 0 & 0 \end{pmatrix}.$$

g_t are generalized to proper holomorphic map $G_t : \Omega^I_{r,s} \to \Omega^I_{2r-1,2s}$, $0 \leq t \leq 1$, given by for $Z = (z_{ij})_{1\leq i\leq r,\, 1\leq j\leq s}$.

(4.3)
$$Z \mapsto \begin{pmatrix} \sqrt{t}z_{11}^2 & \sqrt{t}z_{11}z_{12} & \cdots & \sqrt{t}z_{11}z_{1s} & \sqrt{1-t}z_{11} & z_{12} & \cdots & z_{1s} \\ \sqrt{t}z_{11}z_{21} & \sqrt{t}z_{21}z_{12} & \cdots & \sqrt{t}z_{21}z_{1s} & \sqrt{1-t}z_{21} & z_{22} & \cdots & z_{2s} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \sqrt{t}z_{11}z_{r1} & \sqrt{t}z_{r1}z_{12} & \cdots & \sqrt{t}z_{r1}z_{1s} & \sqrt{1-t}z_{r1} & z_{r2} & \cdots & z_{rs} \\ z_{21} & z_{22} & \cdots & z_{2s} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ z_{r1} & z_{r2} & \cdots & z_{rs} & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Proof of Theorem 1.2. The same method of the proof in Corollary 4.2 can be applied. We omit the proof. $\square$

4.2. 1-parameter family of proper holomorphic maps among Ω_n^{III}.

Consider the proper holomorphic maps $h_0 : \Omega_2^{III} \to \Omega_3^{III}$ and $h_1 : \Omega_2^{III} \to \Omega_3^{III}$ defined by

$$h_0\left(\left(\begin{array}{cc} z_1 & z_2 \\ z_2 & z_3 \end{array}\right)\right) = \left(\begin{array}{ccc} z_1^2 & \sqrt{2}z_1z_2 & z_2^2 \\ \sqrt{2}z_1z_2 & z_1z_2+z_2^2 & \sqrt{2}z_2z_3 \\ z_2^2 & \sqrt{2}z_2z_3 & z_3^2 \end{array}\right)$$

$$h_1\left(\left(\begin{array}{cc} z_1 & z_2 \\ z_2 & z_3 \end{array}\right)\right) = \left(\begin{array}{ccc} z_1^2 & z_1z_2 & z_2 \\ z_1z_2 & z_2^2 & z_3 \\ z_2 & z_3 & 0 \end{array}\right), \ \text{for} \left(\begin{array}{cc} z_1 & z_2 \\ z_2 & z_3 \end{array}\right) \in \Omega_{2,2}$$

which are the restriction of f_0 and f_1 to Ω_2^{III}. These are homotopically equivalent in the target domain Ω_4^{III} by $h_t : \Omega_2^{III} \to \Omega_4^{III}$ which are proper holomorphic maps given by $h_t(Z) = f_t|_{\Omega_2^{III}}(Z)$. Explicitly,
(4.4)

$$h_t : \left(\begin{array}{cc} z_1 & z_2 \\ z_2 & z_3 \end{array}\right) \mapsto \left(\begin{array}{cccc} z_1^2 & \sqrt{2-t}z_1z_2 & \sqrt{1-t}z_2^2 & \sqrt{t}z_2 \\ \sqrt{2-t}z_1z_2 & \frac{2(1-t)}{2-t}z_1z_3+z_2^2 & 2\sqrt{\frac{1-t}{2-t}}z_2z_3 & \sqrt{\frac{t}{2-t}}z_3 \\ \sqrt{1-t}z_2^2 & 2\sqrt{\frac{1-t}{2-t}}z_2z_3 & z_3^2 & 0 \\ \sqrt{t}z_2 & \sqrt{\frac{t}{2-t}}z_3 & 0 & 0 \end{array}\right).$$

We can apply similar method to h_t and a higher version of h_t, we obtain the following:

Corollary 4.4. *There are uncountably many inequivalent proper holomorphic maps from Ω_n^{III} to $\Omega_{n'}^{III}$ with $n' = \frac{1}{2}(n^2 + 3n - 2)$.*

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